

*N. N. Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine
252143 Kiev, Ukraine. 5.25.1998*

POSSIBLE CONSEQUENCES OF CONJECTURAL PERIODICITY OF SPECTRUM OF LATTICE DIRAC OPERATOR

Abstract

Some consequences which follow from the periodicity assumption for spectral density of Wilson–Dirac operator are studied. Such an assumption allows to obtain simple representations for quark propagator, which reveals an important role of $m \leftrightarrow -m$ symmetry. It is argued that this symmetry is restored when the mirror fermion mass m_r tended to infinity. The constrains on zero modes of Wilson–Dirac operator in a toy model approximation are also discussed.

1 Introduction

The spectrum of Dirac operator is closely related to a number of important aspects in finite temperature QCD. The behavior of spectral density $\rho(\lambda)$ around $\lambda \sim 0$ for massless Dirac operator is of particular physical interest being responsible for breaking the chiral symmetry [1]. The low lying eigenvalues of staggered fermion operator are also used to extract the topological content of background gauge field configurations [2].

Despite steady progress, the numerical simulation results for $\rho(\lambda)$ remain inconclusive. As it is known from general theoretical considerations (see e.g. [3, 4, 5]), the eigenvalues of the normalized Dirac operator are located in a compact area. However, as it pointed out in [6] perturbative calculations show even an increase of state density at large λ . Indeed, for $\lambda \gg \Lambda_{QCD}$ the spectral density is regarded to be insensitive to gluon vacuum fluctuations and behaves in the same way as for free fermions:

$$\rho^{free}(\lambda) \sim \frac{N_c \lambda^3}{4\pi^2}. \quad (1)$$

This free term introduces quadratically divergent piece in $\langle \bar{\psi}\psi \rangle$. However, to get a truly non-perturbative quark condensate which is the order parameter of the spontaneous chiral symmetry breaking, this perturbative divergent part should be subtracted. As a result, the quark condensate is related to the region of small λ : $\lambda \sim m \ll \Lambda_{QCD}$ [6].

Some numerical results [7, 8, 9] may be taken as an evidence in favor of compactness of Dirac operator spectrum (see Fig. 2,4,5 and 6 of ref. [7]). Indeed, the spectral density disappears outside certain region Γ_λ [10], that for a particular case may be specified as: $|\lambda| \lesssim 2\Lambda_{QCD}$ (see Fig. 2 of ref. [6]), or for $|\lambda| \lesssim 3/C$ (see Fig. 2 of ref. [11]). The domain wall fermion approach leads to a similar picture (see [12] Fig. 12). In case of (1+1)-dimension it was proved [13, 14, 15], that the spectrum was contained inside the compact area (circle of radius 2 in the complex plane). Moreover, compactness more often than not may be treated as implicit periodicity, so it is hard to avoid the assumption that the spectral density may appear a periodic function of λ . Indeed,

the lattice regularization for Dirac operator leads to

$$D(A) = \gamma^\nu (\partial_\nu - iA_\nu) - m \rightarrow \sum_{\nu=0}^3 D(A)_{x'x}^\nu, \quad (2)$$

where

$$D(A)_{x'x}^\nu = \frac{r - \gamma_\nu}{2} U_\nu(x) \delta_{x,x'-\nu} + \frac{r + \gamma_\nu}{2} U_\nu^\dagger(x') \delta_{x,x'+\nu} - (r + \delta_\nu^0 m a) \delta_{x',x} \quad (3)$$

and $r \equiv \tanh a_\tau m_r$ Wilson parameter. Let us assume that the fermion part of the action $-S_F = \ln \det D$ can be expressed in terms of Polyakov loops $\Omega(\mathbf{x}) = \prod_{t=0}^{N_\tau-1} U_0(t, \mathbf{x})$, as it can be done in numerous particular cases for the gluon part of the action, and $\det D = f(\varphi)$. In this case one may expect that the Dirac matrix $\det D(\varphi)$ (as well as $\langle \bar{\psi}\psi \rangle$) became periodic in φ

$$\det D(A_0) = \det D(A_0 + 2\pi T), \quad (4)$$

with

$$\varphi = \arg \Omega = a_\tau N_\tau A_0 \equiv \frac{A_0}{T}. \quad (5)$$

Therefore, one may suspect that this condition, together with the periodicity along t - axis, may induce the periodicity of D in the complex plane of m and, hence, in eigenvalues λ . In other words it would not be too surprising if

$$\lambda_n(\varphi + 2\pi) = \lambda_n(\varphi) + 2\pi\Theta; \quad \Theta = const. \quad (6)$$

Indeed, if to take the free Dirac operator as an example [11], the spectrum in the phase $\arg \Omega = 0$ is given by: $\lambda^2 = \mathbf{k}^2 + ((2n+1)\pi T)^2$ and $\lambda^2 = \mathbf{k}^2 + ((2n+1/3)\pi T)^2$ for $\arg \Omega = 2\pi/3$.

More rigorous and convincing evidence in favor of the periodicity of Dirac operator spectrum in general case was suggested in [16] (see also [17, 18]). Indeed, Dirac operator can be presented [19] as

$$aD = 1 - \gamma_5 \epsilon(H)), \quad (7)$$

where $\epsilon(H)$ some Hermitian matrix and $H = \gamma_5[1 - \gamma_\mu(\partial_\mu + iA_\mu)]$. Therefore, if D satisfies the Ginsparg-Wilson relation [20]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D, \quad (8)$$

then the operator $\gamma_5\epsilon(H) = 1 - aD$ is unitary

$$(1 - aD)^\dagger (1 - aD) = (1 - a\gamma_5 D\gamma_5) (1 - aD) = 1. \quad (9)$$

This implies that the spectrum of D lies in the complex plane [16, 17, 18] on the circle, $(1 - e^{i\theta})/a$ with $\theta \in [-\pi, \pi]$. Considering the above, a Banks-Casher-like relation [1] will be

$$\langle \bar{\psi}\psi \rangle = \langle D^{-1}(x, x) \rangle \sim a \int_{-\pi}^{\pi} \frac{\rho(\theta)}{1 + am - e^{i\theta}} d\theta. \quad (10)$$

The periodicity of Dirac spectrum $\rho(\theta)$ in θ may be, as a matter of course, treated as a lattice artefact. Indeed, assuming that the major part of eigenstates is concentrated in the area $\theta^2 \lesssim a^2$, [18] after redefinition of the integration variable $\theta \rightarrow sa$, and lifting lattice regularization ($a \rightarrow 0$) one comes to the standard Banks-Casher relation

$$\langle \bar{\psi}\psi \rangle = \langle D^{-1}(x, x) \rangle \sim \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{\rho(sa)}{\frac{1-e^{isa}}{a} + m} ds \simeq \int_{-\infty}^{\infty} \frac{\rho(sa)}{m - is} ds. \quad (11)$$

Though, one should also consider the possibility that e.g., $\lambda^2 \lesssim 1/T^2$, with $\lambda \equiv \theta T$. In this case the spectrum remains invariant under the shift $\lambda \rightarrow \lambda \pm 2\pi T$. The Banks-Casher relation, of course, remains valid in all cases, however, premature lifting the lattice regularization in (11), leaves the conjectural periodicity hidden.

Certainly, the arguments given above, will not suffice as such and we must admit, that the assumption (6) looks much more controversial than the evidences given in favor of compactness. Below we briefly sketch the results [21] obtained in the approximation, where all terms proportional to $1/\xi$ have been discarded¹ in a Hamiltonian limit $a_\tau \rightarrow 0$ ($\xi \rightarrow \infty$). The spectral density computed in such an approximation is actually periodic. This reassures us a bit and allows us to

¹ The anisotropy parameter ξ is defined as $\xi \simeq a_\sigma/a_\tau$, where a_τ and a_σ are temporal and spatial lattice spacings, respectively.

assume that the periodicity may appear to be a general feature of the spectral density of Dirac operator. Hereafter we consider some consequences which follow from such an assumption for a more general case.

2 Toy model approximation

In a toy model approximation the fermion determinant can be computed analytically and presented in a closed form. We choose the form suggested in [22] (see also [23]) for the fermion part of action

$$-S_F = n_f \left(a^3 \sum_x \bar{\psi}_x D_{x'x}^0 \psi_x + \frac{a^3}{\xi} \sum_{n=1}^d \sum_x \bar{\psi}_x D_{x'x}^n \psi_x \right), \quad (12)$$

where n_f refers to the number of flavors² and $D_{x'x}^\nu$ are given by (3).

Inasmuch as in the suggested approximation we omit the terms of $1/\xi$ order, the 'toy' action will be simply $a^3 \sum_x \bar{\psi}_x D_{xx'}^0 \psi_{x'}$. The projectors $\frac{1 \pm \gamma_0}{2}$ divide the bispinors ψ into two components $\psi^{(\pm)} = \frac{1 \pm \gamma_0}{2} \psi$, each including only one two-component spinor. Therefore, taking into account that $(\delta_{x-\hat{\mu},x'})^\dagger = \delta_{x,x-\hat{\mu}'} = \delta_{x+\hat{\mu},x'}$, and presenting

$$\frac{r \pm \gamma_0}{2} = \frac{1 + \gamma_0}{2} \frac{r \pm 1}{2} + \frac{r \mp 1}{2} \frac{1 - \gamma_0}{2}, \quad (13)$$

we may rewrite (12) as

$$-S_F = \sqrt{1 - r^2} \left(\bar{\psi}_{x'}^{(+)} \Delta_{x'x}^\dagger \psi_x^{(+)} + \bar{\psi}_{x'}^{(-)} \Delta_{x'x} \psi_x^{(-)} \right), \quad (14)$$

where

$$\Delta_{xx'} = \delta_{\mathbf{x}\mathbf{x}'} \left(\frac{e^{a_\tau m_r} U_0(\mathbf{x}, t)}{2} \delta_{t', t-1} - \frac{e^{-a_\tau m_r} U_0^\dagger(\mathbf{x}, t')}{2} \delta_{t', t+1} - \frac{ma_\tau + r}{\sqrt{1 - r^2}} \delta_{t', t} \right). \quad (15)$$

By gauge transformation all $U_0(\mathbf{x}, t)_{\alpha\nu}$ matrices may be diagonalized simultaneously: $U_0(\mathbf{x}, t)_{\alpha\nu} = \delta_{\alpha\nu} U_0(\mathbf{x}, t)_{\alpha\alpha}$, therefore $\Delta_{xx'}$ in (15) is simply a set of N matrices $N_\tau \times N_\tau$. Hopefully, the computation of $\det \Delta$, in the considered extremely simple case can be done straightforwardly and after the integration over $\psi_x^{(\pm)}$ we get

$$-S_F^{eff} = n_f \sum_\alpha \ln \det \Delta_\alpha^\dagger \Delta_\alpha = n_f \ln \prod_{\alpha=1}^N (z_r - \cos \varphi_\alpha) \left(z - (-1)^{N_\tau} \cos \varphi_\alpha \right), \quad (16)$$

² We consider degenerate quarks.

with

$$z = (-1)^B \cosh \frac{m}{T}; \quad z_r = (-1)^B \cosh \frac{m+2m_r}{T}, \quad (17)$$

where $B = 0$ for periodic and $B = 1$ for antiperiodic border conditions on fermion fields in a temporal direction. Thereafter we shall consider the standard case $B = 1$ and even values N_τ , because the generalization on $B = 0$ and odd N_τ is quite evident.

It is easy to see from (16) and (17) that $\langle \bar{\psi} \psi \rangle$ may be presented as

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= 2n_f \sinh \frac{m}{T} \left\langle \frac{1}{\cosh \frac{m}{T} + \cos \varphi} \right\rangle \\ &\quad + 2n_f \sinh \frac{m+2m_r}{T} \left\langle \frac{1}{\cosh \frac{m+2m_r}{T} + \cos \varphi} \right\rangle, \end{aligned} \quad (18)$$

where m_r is the mirror fermion mass connected to Wilson parameter by the relation $a_\tau m_r = \operatorname{arctanh} r$.

3 General case

Recall that the fermion Green's function in the external gauge field is given by

$$\langle \bar{\psi}(y) \psi(x) \rangle = \sum_n \frac{u_n^\dagger(y) u_n(x)}{m - i\lambda_n}, \quad (19)$$

where $\lambda_n(x)$ eigenvalues of the *massless* Dirac operator and u_n are corresponding eigenfunctions. The spectrum of the massless Dirac operator is discrete and enjoys the chiral symmetry. In particular, for any eigenfunction $u_n(x)$, the function

$$\tilde{u}_n = \gamma^5 u_n \quad (20)$$

is the eigenfunction as well, with the eigenvalue

$$\tilde{\lambda}_n = -\lambda_n. \quad (21)$$

Therefore

$$\rho(\lambda) = \rho(-\lambda) \quad (22)$$

and eigenfunctions occur in pairs with opposite eigenvalues, except for zero modes³. In the limit $V \rightarrow \infty$, the level spectrum becomes dense and we can trade the sum in (19) for the integral and get

$$\langle \bar{\psi} \psi \rangle = 2m \int_0^\infty \frac{\rho(\lambda)}{m^2 + \lambda^2} d\lambda, \quad (23)$$

where

$$\rho(\lambda) = \frac{1}{V} \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle. \quad (24)$$

In case when spectral density is compact and periodic with the some period Θ

$$\rho(\lambda) = \rho(\lambda + 2k\pi\Theta) \quad (25)$$

we can write

$$\langle \bar{\psi} \psi \rangle_m = m \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} \frac{\rho(\lambda)}{m^2 + \lambda^2} d\lambda \quad (26)$$

or

$$\langle \bar{\psi} \psi \rangle_m = m \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \frac{\rho(\lambda)}{m^2 + (\lambda + 2\pi k\Theta)^2} d\lambda. \quad (27)$$

Presenting ([24] 5.1.25(3))

$$\frac{m}{\Theta} \sum_{k=-\infty}^{\infty} \frac{1}{\left(\frac{\lambda}{\Theta} + 2k\pi\right)^2 + \left(\frac{m}{\Theta}\right)^2} = \frac{\sinh \frac{m}{\Theta}}{\cosh \frac{m}{\Theta} - \cos \frac{\lambda}{\Theta}}, \quad (28)$$

we get

$$\langle \bar{\psi} \psi \rangle = \sinh \frac{m}{\Theta} \times \int_0^{2\pi} \frac{\rho(\varphi\Theta)}{\cosh \frac{m}{\Theta} - \cos \varphi} d\varphi. \quad (29)$$

So we come to the conclusion that (18) is nothing but a particular case of (29) for the periodic spectral density $\rho(\lambda)$ with the period $\Theta = T$. It easy to check that it leads to the expression for $\langle \bar{\psi} \psi \rangle$ that coincides with (18) obtained in a toy model approximation.

³ Generally, symmetry (22) does not mean chiral invariance. Indeed, the eigen functions of Dirac operator are the same for $m = 0$ and $m \neq 0$. Therefore, the symmetry (22) is preserved for arbitrary m , though only massless Dirac operator is invariant under transformation (20).

If one claims

$$\rho(\lambda) = \rho(\lambda_1; \lambda_2 \lambda_3) = \rho(\lambda_1 + 2\pi k_1 T; \lambda_2 + 2\pi k_2 T; \lambda_3 + 2\pi k_3 T) \quad (30)$$

and

$$\sum_{\alpha=1}^3 \lambda_\alpha = 2\pi k_0 T, \quad (31)$$

where all k_j are integer, one can easily get an expression for $\langle \bar{\psi} \psi \rangle$ which coincides with the result obtained in [21] for $SU(3)$ gauge group case⁴. So we come to the conclusion that the 'toy' model gives the results that may appear reasonable in the general case ($\xi \sim 1$) if periodicity assumption happens to be allowable.

4 Some formal consequences of discrete symmetries

A very formal expression of the symmetry (20), (22) can be found if we take into account the following. As it is known the transformation under CPT reversal can be presented as a combination of the unitary transformation⁵ $\psi(t, \mathbf{x}) \rightarrow \gamma_5 \psi(t, \mathbf{x})$ and inversion $\psi(t, \mathbf{x}) \rightarrow \psi(-t, -\mathbf{x})$. Under unitary transformation as well as under inversion the massless Dirac operator \mathfrak{D} only changes the sign. Therefore, the result of the unitary transformation of $\mathfrak{D} - m$ can be mimicked by the formal substitution $m \rightarrow -m$. It is easy to check that $\rho(\lambda) \leftrightarrow \rho(-\lambda)$ invariance also leads to $m \leftrightarrow -m$ symmetry

$$\Sigma(m) = \langle \bar{\psi} \psi \rangle = \int_{-\infty}^{\infty} \frac{\rho(\lambda)}{m - i\lambda} d\lambda = \int_{-\infty}^{\infty} \frac{\rho(-\lambda)}{m - i\lambda} d\lambda = -\Sigma(-m). \quad (32)$$

To be specific, we confine ourselves to the case of $SU(2)$ gauge group. Taking into account that

$$\frac{1}{\cosh x \pm \cos \frac{\varphi}{2}} = 2 \sum_{j=0}^{\infty} (\mp 1)^{2j} \chi_j(\varphi) e^{-(2j+1)|x|}, \quad (33)$$

⁴ Of course, specific form of periodicity in (30) was chosen intentionally to achieve such coincidence.

⁵ This invariance under chiral rotation $\psi \rightarrow \exp\{i\gamma_5 \theta\} \psi$ rotated by an angle $\theta = \pi$ is, in fact, the remnant of chiral symmetry of the massless theory.

where $\chi_j(\varphi) = \frac{\sin(2j+1)\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}$ are the characters of irreducible representations of $SU(2)$ group, we can obtain from (18)

$$\begin{aligned}\langle\bar{\psi}\psi\rangle &= 4n_f \sinh \frac{m}{T} \sum_{j=0}^{\infty} e^{-(2j+1)\left|\frac{m}{T}\right|} (-1)^{2j} \langle\chi_j\rangle + \\ &\quad 4n_f \sinh \frac{m+2m_r}{T} \sum_{j=0}^{\infty} e^{-(2j+1)\left|\frac{m+2m_r}{T}\right|} (-1)^{2j} \langle\chi_j\rangle,\end{aligned}\tag{34}$$

so the breakdown of the symmetry (22) (and consequently (32)) by the second term in (34) for any $m_r \neq 0$ became apparent. Moreover, S_F posses an undesirable invariance under the interchange

$$m \leftrightarrow -m - 2m_r.\tag{35}$$

This invariance is inherent neither for Dirac operator nor for the spectrum (23). In the limit $a_\tau \rightarrow 0$ such symmetry is cognate to the invariance under the hopping parameter interchange: $K \rightarrow -K$, which plays an important role in the emergence of conjectural Aoki phases, widely discussed in literature (see e.g., [25, 26, 27]).

Although the mirror fermion input does not disappear in $a_\tau \rightarrow 0$ limit, almost all mirror terms exponentially vanish for $\frac{m_r}{T} \rightarrow \infty$. The only surviving term $j = 0$, however, introduces finite contributions into $\langle\bar{\psi}\psi\rangle$ which breaks $m \rightarrow -m$ symmetry. Let us consider a very special case $m_r \rightarrow \infty$, in other words, put exactly $r = 1$ in the action (12) *from the beginning*.

It is easy to see that in this case

$$\Delta_{x',x} = \delta_{\mathbf{x}'\mathbf{x}} (U_0(\mathbf{x}, t) \delta_{t,t'-1} - (ma_\tau + 1) \delta_{t,t'}) \simeq \delta_{\mathbf{x}'\mathbf{x}} (U_0(\mathbf{x}, t) \delta_{t,t'-1} - e^{a_\tau m} \delta_{t,t'}),\tag{36}$$

therefore

$$\det \Delta_{x',x} = \prod_{\alpha} (e^{\frac{m}{T}} + e^{i\varphi_{\alpha}}),\tag{37}$$

and

$$\exp \{-S_F\} = n_f \det \Delta \det \Delta^\dagger = n_f \prod_{\alpha} \left(\cosh \frac{m}{T} + \cos \varphi_{\alpha} \right)\tag{38}$$

where $e^{i\varphi_\alpha}$ are eigenvalues of the Polyakov matrix $\Omega_{\alpha\beta} = \delta_{\alpha\beta}e^{i\varphi_\alpha}$. In this case (at least in zero order in $1/\xi$) mirror fermion term does not appear at all and terms leading in $1/\xi$ are invariant under the change $m \rightarrow -m$. The mirror fermion contributions in (37) are shifted into the area of infinite m_r and undesirable symmetry (35) is expired.

It is interesting to note that, after 'harmless' substitution $1 + a_\tau m \simeq e^{a_\tau m}$ the periodicity in λ may appear in a general case. Indeed, we may expect that in the limit ($a_\tau \rightarrow 0; N_\tau \rightarrow \infty$) the mass variable will survive only in combination $(e^{a_\tau m})^{N_\tau q} = e^{\frac{m}{T}q}$, where q is some dimensionless number. Thereby we get periodicity $m \rightarrow m + 2\pi iT/q$. Taking into account that eigenvalues λ of the operator $\mathfrak{D} = \gamma^\nu (\partial_\nu - iA_\nu)$ defined by $\det(\mathfrak{D} - i\lambda) = 0$ are related to eigenvalues λ_m of the operator $\mathfrak{D} - m$ by $\lambda_m = \lambda + im$ we also get (as can be seen e.g., from (23)) periodicity in λ

$$\lambda \rightarrow \lambda + 2\pi T/q. \quad (39)$$

Recall that for finite N_τ, a_τ and m_r one may write in a toy model approximation [21]

$$-S_F^{eff} = \sum_\alpha \ln \det \Delta_\alpha^\dagger \Delta_\alpha = \sum_\alpha \Xi_\alpha^* \Xi_\alpha, \quad (40)$$

where

$$\Xi_\alpha(m') = \cosh(N_\tau \operatorname{arcsinh}(m_r a_\tau + m a_\tau)) + \frac{1}{2} e^{\frac{m_r}{T}} \Omega_\alpha + \frac{1}{2} e^{-\frac{m_r}{T}} \Omega_\alpha^*. \quad (41)$$

Thereby at finite a_τ we may expect only approximate periodicity in λ . That evidently differs from the case of $m_r \rightarrow \infty$, which leads to (38).

5 Zero modes

Among the non-perturbative properties of quarks, those associated with the topology of the gauge fields occupy a special place. An important insight can be gained about the topological content of gauge field configurations in finite temperature QCD by computing the low lying modes of the Dirac operator. In the continuum the Dirac operator of massless fermions in a smooth background gauge field with non-zero topological charge has zero eigenvalues. The corresponding eigenfunctions are chiral and the number of left- and right-handed zero modes are related to the topological

charge of the gauge configuration: $n_L - n_R = \nu$ [28]. Besides, the study of the Dirac operator spectrum near $\lambda = 0$ is of fundamental importance for the understanding of chiral symmetry breaking in gauge theories.

Unfortunately, the lattice formulation also introduces artifacts and there is no exact index theorem in this context. Instead, what is observed with the actions currently used is the “zero mode shift” phenomenon [29] (i.e. modes with small but non-zero eigenvalue and not exactly chiral). Typical gauge field configurations contributing to the path integral are not smooth and, in addition, lattice regularization breaks some other conditions of the index theorem as well: the topological charge of coarse gauge configurations is not properly defined and Dirac operator breaks chiral symmetry in an essential way [30]. Unlike the continuum, there is no division of lattice gauge fields into disconnected classes since lattice gauge fields are characterized by group elements on the links of the lattice and all link variables can be deformed to unity [31].

A noticeable progress in overcoming such difficulties was recently achieved . In particular, as it is argued in [30], the lattice QCD in fixed point formulation solves most of mentioned problems and in such context index theorem seems to be valid on the lattice. The lattice version of index theorem, which connects the small real eigenvalues of Wilson-Dirac operator and the geometrical definition of topological charge, was suggested in [32].

Reliable zero mode separation in numerical computations still remains a serious problem [31], so we hope that even crude analytical models, which allow to estimate $\rho(\lambda)$ behavior in the area of small λ , may appear useful. Notwithstanding the abrupt fall-off and vanishing of $\rho(\lambda)$ at $\lambda = 0$ measured in [33] may be a finite volume effect [6], the present data par excellence do not confront the suggestion that $\rho(0)$ is very small and compatible with zero. As it was shown in [34] the spectral density decreases as λ approaches zero, and the larger is n_f — the larger is the effect. The scenario when $\rho(0)$ hits zero at $n_f \geq 5$ is quite probable.

Dirac spectrum should be particularly sensitive to the effects of quark loops since fermion determinant for small quark mass strongly suppresses gauge configurations with small Dirac eigenvalues.

Convincing arguments that spectral density disappears at $\lambda = 0$ are given in [34]. As it is pointed out there, the weight in averaging involves a fermion determinant factor

$$[\text{Det}(\mathfrak{D} - m)]^{n_f} = \left[m^{|\nu|} \prod_{\lambda_n > 0} (\lambda_n^2 + m^2) \right]^{n_f}. \quad (42)$$

To be more specific we turn to the simple case of toy model approximation where fermion part of action can be totally separated from the gluon one and in accordance with (38) incorporated into the measure by interchanging $d\mu[\varphi_\alpha(x)] \rightarrow d\tilde{\mu}[\varphi_\alpha(x)]$ with

$$d\tilde{\mu}[\varphi_\alpha(x)] = \prod_{\alpha=1}^N \left(\cosh \frac{m}{T} + \cos \varphi_\alpha(x) \right)^{n_f} d\mu[\varphi_\alpha(x)], \quad (43)$$

with

$$d\mu = \prod_{n>m}^N 2 \sin^2 \frac{\varphi_n - \varphi_m}{2} \delta \left(\sum_{k=1}^N \varphi_k \right) \prod_{k=1}^N \frac{d\varphi_k}{2\pi}; \quad (44)$$

So the measure (43) has a set of n_f -multiple zeroes, which evidently corresponds to the zeroes of (42). The effective action $S_G^{eff} = S_G^{eff}[\varphi_\alpha(x)]$ for gluon field (e.g. the most popular S_G^{eff} , suggested in [35, 36]) remains finite and scarcely able to influence either multiplicity or locus of the zeroes. Zero modes constitute no exception and are completely defined by the zeroes of (43).

As it is easy to see from (36), the eigenvalues of the matrix $\Delta_{x',x}$, defined by $\det(\Delta - i\lambda) = 0$, can be easily found from (37) after the simple shift $m \rightarrow m - i\lambda$

$$\lambda = -im + T(\varphi_\alpha \pm (2n+1)\pi) \quad (45)$$

and one can easily show that the eigenvalues of the matrix $\Delta_{x',x}^\dagger$ differs from (45) only by the sign of the real part.

The expression (45) plainly shows that real modes appear only in the limit $m \rightarrow 0$ and are placed at $T(\varphi_\alpha \pm (2n+1)\pi)$. It should also be noted that zero modes created by the fields $\psi^{(\pm)} = \frac{1 \pm \gamma_0}{2} \psi$ are divided into pairs $\pm\lambda$ as it is for $\psi^{(L)} = \frac{1-\gamma_5}{2} \psi$ and $\psi^{(R)} = \frac{1+\gamma_5}{2} \psi$. Although the total numbers of the pairs $n_L + n_R$ and $n_+ + n_-$ are equal, the corresponding sets of eigenvectors, of course, do not obligatory coincide, so $\nu = n_L - n_R$ and $\nu' = n_+ - n_-$ may differ. Therefore, the topological nature of ν' , if any, needs special consideration.

It is not incurious that the spectral density can be treated simply as a result of integration of the *QCD* action over all variables φ_x but one φ or the integration over the surface with the fixed average $\chi(\varphi) = \frac{1}{V} \sum_x^V \chi(\varphi_x)$. For example, in case of $SU(2)$ gauge group one may write

$$\rho\left(\frac{\varphi}{2}\Theta\right) = n_f \left\langle \delta\left(\chi(\varphi) - \frac{1}{V} \sum \chi(\varphi_x)\right) \right\rangle \frac{d\mu(\varphi)}{d\varphi} = n_f \exp\{-S_{(1)}(\varphi)\} \frac{2}{\pi} \sin^2\left(\frac{\varphi}{2}\right). \quad (46)$$

The relation between λ and the phases of Polyakov loop eigenvalues φ_α makes it obvious that λ are not fixed numbers, the dependence λ on the parameters of the theory are defined not only by the fermion part of action, but by the gluon part as well. In particular, zero modes are defined by the relative number of Polyakov loops with $\cos \varphi_\alpha \simeq -\cosh \frac{m}{T} \simeq -1$. It is obvious that when the gauge fields are not fixed ($\varphi_\alpha(\mathbf{x}) \neq \text{const}$), the behavior of the corresponding $\det \Delta(\varphi)$ in the limit $m \rightarrow 0$ is generally different for different $\varphi_\alpha(\mathbf{x})$. Therefore, (46) does not result in an integer value for index ν and we obtain the probability distribution, $p(\nu)$, which is defined by $S_{(1)}(\varphi)$.

However, we can compute the extreme value of $\nu : \nu_{\max} = |\max\{\nu\}|$.

Taking into account the condition $\sum_\alpha \varphi_\alpha = 2\pi n$ we get $\nu_{\max} = \left(4 + \frac{2}{n_f}\right) \left[\frac{N}{2}\right]$, that both for $SU(2)$ and $SU(3)$ gives the same result $\nu_{\max} = 4 + \frac{2}{n_f}$.

In a series of papers [37, 38, 33, 39], substantial new insight was added to this issue. Their central assertion is that the spectral density of Dirac operator very close to the origin $\lambda = 0$ should be universal, depending only on the symmetries in question. One of the amazing consequences of this conjecture is that the spectral density of Dirac operator near the origin need not be computed in full *QCD*, but can be extracted from simple models.

We cannot exclude that simple analytical structure of S_F as in (38) will not persist in the general case, moreover, such simplicity is in conflict with numerical results obtained in [14]. If, nevertheless, one ventures to apply universality arguments [37, 38, 33, 39] to our model, it will lead to $\rho(\lambda) \sim \left(\frac{\lambda}{T}\right)^\nu$ with $\nu = \nu(g, T)$ and $\nu_{\max} = 4 + \frac{2}{n_f}$. Thus, the configurations with small eigenvalues are effectively suppressed, and the larger n_f is, the more prominent is the suppression. Therefore, in the computation of averages their contribution is small. It is true even for $\langle \bar{\psi}\psi \rangle$

because $\bar{\psi}\psi$ has poles only in the points where fermion determinant turns to zero.

6 Conclusions

We have presented some evidence in favor of periodicity suggestion for the spectral density of Dirac operator on the lattice. Indeed, such periodicity appears in the limit $a_\tau \rightarrow 0$ in a toy model approximation [21] and in our opinion there are reasons to believe that it still persists in the general case.

In this paper we take advantage of the fact that the chiral invariance of the *massless part* of Dirac operator leads to formal symmetry $m \leftrightarrow -m$. This symmetry is of importance in studying *QCD* phase structure in $(\mu; T; m)$ space [40]. It appears desirable to make such symmetry explicit on finite size lattice. In case of infinitely heavy mirror fermion ($r = 1$), the symmetry $m \leftrightarrow -m$ is restored. Moreover, the periodicity of spectral density has also become explicit for $r = 1$ already at finite a_τ .

Besides, we suggest a simple way to relate the spectral density $\rho(\lambda)$ to the quasiaverage $S^{(1)}$ of *QCD* action with Polyakov loop matrices with $\arg \Omega_\alpha(\bar{x}) = \varphi_\alpha(\bar{x})$ at some point \bar{x} being fixed. As it can be anticipated (see e.g.,[41]) the spectrum of Dirac operator is not invariant under $Z(N)$ transformations and the results are different in the phases $\arg \Omega = 0$ and $\arg \Omega = \frac{2\pi}{3}$. Nonetheless, for $\arg \Omega = \frac{2\pi}{3}$ and $\arg \Omega = -\frac{2\pi}{3}$ the results coincide, which reflects C - symmetry ($\varphi_\alpha \leftrightarrow -\varphi_\alpha$) [42] at zero density $\mu = 0$.

In this paper, we have taken a modest first step towards analytical investigation of Dirac spectral density characteristics. Apart from that there are many other questions which can be answered with the low lying eigenvalues and eigenvectors. The correlations between eigenvalues, the importance of which was proved in [31], is one of them. However, it is much more demanding a task than we can afford in such letter and we shall try to discuss it elsewhere

References

- T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
- R.G. Edwards, U.M. Heller and R. Narayanan, e-Print Archive: hep-lat/9801015, '*The hermitian Wilson-Dirac operator in smooth $SU(2)$ instanton backgrounds*'.
- M. Luscher, DESY-93-148, Nucl.Phys. **B418** (1994) 637, e-Print Archive: hep-lat/9311007, '*A NEW APPROACH TO THE PROBLEM OF DYNAMICAL QUARKS IN NUMERICAL SIMULATIONS OF LATTICE QCD*'.
- B. Bunk, K. Jansen, B. Jegerlehner, M. Luscher, H. Simma, R. Sommer, Nucl.Phys.Proc.Suppl. **42** (1995) 49, e-Print Archive: hep-lat/9411016, '*A NEW SIMULATION ALGORITHM FOR LATTICE QCD WITH DYNAMICAL QUARKS*'.
- K. Jansen, C. Liu, H. Simma, D. Smith, Nucl.Phys.Proc.Suppl. **53** (1997) 262, e-Print Archive: hep-lat/9608048, '*LOW LYING EIGENVALUES OF THE WILSON-DIRAC OPERATOR*'.
- A.V. Smilga, e-Print Archive: hep-th/9503049, '*Chiral symmetry and spectrum of Euclidean Dirac operator in QCD*'.
- T. Kalkreuter, Comput.Phys.Commun. **95** (1996) 1, e-Print Archive: hep-lat/9509071, '*STUDY OF CULLUM'S AND WILLOUGHBY'S LANCZOS METHOD FOR WILSON FERMIONS*'.
- T. Kalkreuter, Nucl.Phys.Proc.Suppl. **42** (1995) 882, e-Print Archive: hep-lat/9410006, '*SPECTRUM OF THE DIRAC OPERATOR AND INVERSION ALGORITHMS WITH DYNAMICAL STAGGERED FERMIONS*'.
- T. Kalkreuter, Phys.Rev. **D51** (1995) 1305, e-Print Archive: hep-lat/9408013, '*SPECTRUM OF THE DIRAC OPERATOR AND MULTIGRID ALGORITHM WITH DYNAMICAL STAGGERED FERMIONS*'.
- J.C.Osborn, J.J.M. Verbaarschot, e-Print Archive: hep-ph/9803419, '*THOULESS ENERGY AND CORRELATIONS OF QCD DIRAC EIGENVALUES*'.
- M.A. Stephanov, Phys.Lett. **B375** (1996) 249, e-Print Archive: hep-lat/9601001, '*CHIRAL SYMMETRY AT FINITE T, THE PHASE OF THE POLYAKOV LOOP AND THE SPECTRUM OF THE DIRAC OPERATOR*'.
- K. Jansen, DESY-94-188, Phys.Rept. **273** (1996) 1, e-Print Archive: hep-lat/9410018, '*DOMAIN WALL FERMIONS AND CHIRAL GAUGE*'.
- C.R. Gattringer, I. Hip, and C.B. Lang, Nucl.Phys. **B508** (1997) 329, e-Print Archive: hep-lat/9707011.
- W. Bardeen, A. Duncan, E. Eichten and H. Thacker, e-Print Archive: hep-lat/9710020, ' *QED_2 as a testbed for interpolations between quenched and full QCD*'.
- W. Bardeen, A. Duncan, E. Eichten and H. Thacker, e-Print Archive: hep-lat/9806002.
- H. Neuberger, e-Print Archive: hep-lat/9801031, '*More about exactly massless quarks on the lattice*'.
- R. Narayanan, hep-lat/9802018, '*Ginsparg-Wilson relation and the overlap formula*'.

- K. Splittorff and A.D. Jackson, e-Print Archive: hep-th/980518, '*The Ginsparg-Wilson relation and local chiral random matrix theory*'
- H.Neuberger, Phys.Lett. **B417** (1998) 141, e-Print Archive: hep-lat/9707022, '*Exactly massless quarks on the lattice*'.
- P.H. Ginsparg and K.G. Wilson, Phys. Rev. **D25** (1982) 2649.
- V.K. Petrov. e-Print Archive: hep-lat/9804014, '*ANALYTICAL STUDY OF FERMION DETERMINANT AND CHIRAL CONDENSATE BEHAVIOR AT FINITE TEMPERATURES IN TOY MODEL APPROXIMATION*'.
- P. Hasenfratz, F. Karsch, Phys.Lett. **125B** (1983)308, '*CHEMICAL POTENTIAL ON THE LATTICE*'.
- I. Bender, H.J. Rothe, W. Wetzel, I.O. Stamatescu, HD-THEP-92-46, Z.Phys. **C58** (1993) 333, '*LATTICE FERMIONS AT NONZERO TEMPERATURE AND CHEMICAL POTENTIAL*'.
- A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, '*Integrals and Series*', Nauka(Moskow)1981, Gordon and Breach (New York),1992.
- S. Aoki, Phys. Rev. **D30** (1984) 2653, '*NEW PHASE STRUCTURE FOR LATTICE QCD WITH WILSON FERMIONS*'.
- S. Aoki, T. Kaneda and A. Ukawa, Phys.Rev. **D56** (1997) 1808, e-Print Archive: hep-lat/9612019, '*STRUCTURE OF CRITICAL LINES IN QUENCHED LATTICE QCD WITH THE WILSON QUARK ACTION*'.
- S. Sharpe, R. Singleton, e-Print Archive: heplat/9804028.*Spontaneous Flavor and Parity Breaking with Wilson Fermions*.
- M. Atiyah and I. M. Singer, Ann. Math. **93** (1971) 139.
- J.C.Vink, Phys. Lett. **210B** (1988) 211.
- P. Hasenfratz, V. Laliena, F. Niedermayer, e-Print Archive: hep-lat/9801021, '*THE INDEX THEOREM IN QCD WITH A FINITE CUTOFF*'.
- J.B. Kogut, J.-F. Lagaë and D.K. Sinclair, e-Print Archive: hep-lat/9801020, '*Topology, fermionic zero modes and flavor singlet correlators in finite temperature QCD*', J.B. Kogut, J.-F. Lagaë and D.K. Sinclair, e-Print Archive: hep-lat/9709067, Nucl.Phys.Proc.Suppl. **63** (1998) 433, '*Topology, fermionic zero modes and flavor singlet correlators in finite temperature QCD*'. J.B. Kogut, J.-F. Lagaë and D.K. Sinclair, e-Print Archive: hep-lat/9806001, '*THERMODYNAMICS OF LATTICE QCD WITH 2 QUARK FLAVOURS: CHIRAL SYMMETRY AND TOPOLOGY*'.
- P. Hernández, e-Print Archive: hep-lat/9801035, '*On the Index Theorem for Wilson Fermions*'
- J.J.M. Verbaarschot, *Acta. Phys. Pol. B25* (1994) 133.
- A.V. Smilga. TPI-MINN-96-23, Phys.Rept. **291** (1997) 1, e-Print Archive: hep-ph/9612347, '*PHYSICS OF THERMAL QCD*'.
- F. Green and F. Karsch, Nucl. Phys. **B238** (1984) 297.

M. Ogilvie, Phys. Rev. Lett. **52** (1984) 1369, 'AN EFFECTIVE SPIN MODEL FOR FINITE TEMPERATURE QCD'.

E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306.

J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852.

J.J.M. Verbaarschot, Nucl. Phys. **B426** (1994) 559, J.J.M. Verbaarschot, Phys. Lett. **B368** (1996) 137, J.J.M. Verbaarschot, Nucl.Phys.Proc.Suppl. **53** (1997) 88.

M. A. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov, and J. J. M. Verbaarschot, e-Print Archive: hep-lat/9804290, 'On the Phase Diagram of QCD'.

S. Chandrasekharan and N. Christ, Nucl. Phys. **B** (Proc. Suppl.) **47** (1996) 527, e-Print Archive: hep-lat/9509095, 'Dirac Spectrum, Axial Anomaly and the QCD Chiral Phase Transition'.

O.A. Borisenko, V.K. Petrov, G.M. Zinovev, Phys.Lett. **B221** (1989) 155, 'ORDER PARAMETER PROBLEM IN LATTICE QCD THERMODYNAMICS'.